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The Payload Capabilities of Ion Propulsion Rocket Systems*

Robert H. Fox†

Abstract

The relationship between booster performance, ion propulsion system characteristics, mission, and payload is discussed for the simple case of zero gravitational field. It is shown that appreciable increases in payload can be obtained by varying the exhaust velocity in the appropriate way.

Introduction

There has been much discussion in the literature in recent years on ion rocket propulsion. Many of the problems, such as specific power limitations, thrust programming, flight trajectories, and the like have been examined quite critically. By now it is well recognized that this class of devices possesses inherently low acceleration but also potentially low mass ratios for rather difficult missions. It is not clear, however, how such a system might compare with nuclear rockets using hydrogen propellant. To make such a comparison one must analyze both concepts in some detail and find the areas of superior performance for each. This is much too ambitious a task at present since the limiting problem areas are not understood in detail for either case. Such understanding can arise only after considerable hardware development has taken place. One can make considerable headway with respect to the ion rocket, however, on the basis of some rather simple ideas.

We shall consider only the simple case of a field-free space. The nature of the specific energy and specific power limitations, together with their effect upon the relation between payload "cost" and the desired mission, will be investigated. The results will illustrate the problems involved and serve as a guide for the real problems of flight in a gravitational field.

From the discussion that follows we find that the nature of the mission and the characteristics of the propulsion plant determine the values for mass ratio and exhaust velocity which lead to maximum payload. Except for very easy missions (short distances, long mission time, or low final velocity) we see that the optimum mass ratio is not close to unity. This result is rather interesting since the ion rocket is frequently thought of as a low mass-ratio rocket because of the potentially very high exhaust velocity.

The Energy Limitation

Since a nuclear reactor does not at present represent an infinite energy source one must consider the specific energy (total energy available for the exhaust/total propulsion machinery mass) limitation and its effect upon the payload for a given mission. For very ambitious missions one may find that performance cannot be improved simply by adding more propellant mass, since this may imply more nuclear reactor mass to provide the extra energy required. Examples will be given later which demonstrate that this is a real limit. The energy-limited cases correspond to missions for which the coasting time is large compared to the powered-flight time. In this case, the velocity at the time of cessation of thrust is the prime index of performance.

The problem may be stated briefly: Given a rocket of initial mass M_1, fuel mass $M_f < M_1$, and energy E to be distributed over M_f, which is the maximum value of the final velocity V? Let us assume that the exhaust velocity may be programmed according to the value of m, the total amount of propellant consumed prior to a given moment. Then the rocket equation is:

$$dV = \frac{v(m)dm}{M_1 - m}. \tag{1}$$

Integrating,

$$V_2 - V_1 \int_0^{M_f} \frac{v(m)\,dm}{M_1 - m} \tag{2}$$

$v(m)$ is an arbitrary function. We want to maximize $V_2 - V_1$ subject to the auxiliary condition

$$\frac{1}{2}\int_0^{M_f} v^2(m)\,dm = E \tag{3}$$

The solution of the variational problem posed by (2), (3) has been obtained previously.[1]

$$v(m) = \sqrt{\mu}\,\frac{M_2}{M_1 - m}\,\bar{v} \tag{4}$$

where

$$v_0 = \sqrt{\frac{2E}{M_f}}$$

$$M_2 = M_1 - M_f$$

$$\mu = M_1/M_2.$$

Substituting (4) into (2) the result is:

$$V_2 - V_1 = \bar{v}\,\frac{(\mu - 1)}{\sqrt{\mu}}. \tag{5a}$$

Compare this with the expression obtained for constant exhaust velocity,

$$V_2 - V_1 = \bar{v}\ln\mu \tag{5b}$$

Plots of (5a), (5b) are shown in Figure 1.

* Work done under the auspices of the U. S. Atomic Energy Commission.

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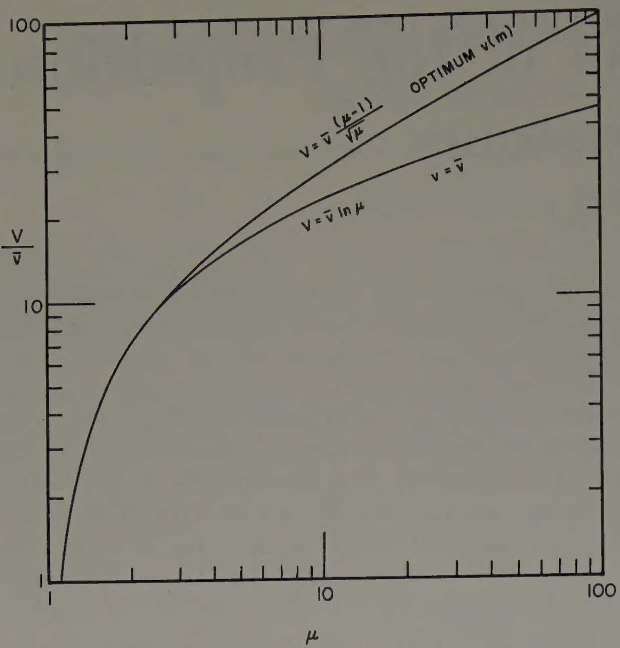

FIG. 1. Dependence of V upon the average velocity $\bar{v}$ and the mass ratio μ for constant exhaust velocity and for optimum exhaust velocity programming.

We can now proceed to an analysis of the "cost" per unit payload mass and its relation to booster performance, difficulty of the mission, specific energy of the ion rocket power plant, and choice of exhaust velocity program. For this purpose, let us assume that the cost is proportional to the total system structural mass (booster structure plus ion rocket propulsion system). This implies that no components are used for more than one trip. The major fraction of this cost is then associated with the booster structure (except perhaps for a good nuclear heat-exchanger rocket with hydrogen propellant) so that the cheapest system results when the burnout velocity for the booster is just sufficient to place the ion rocket into a satellite orbit.

The important figure of merit for the booster is then

$$\beta = \frac{M_1}{M_b}. \quad M_1 = \text{initial ion rocket mass} \qquad M_b = \text{booster dry mass} \tag{6}$$

For the ion rocket second stage,

$$M_1 = M_f + M_p + M_L. \quad M_f = \text{propellant}, \quad M_p = \text{propulsion machinery}, \quad M_L = \text{payload} \tag{7}$$

Then

$$\mu = \frac{M_1}{M_2} = \frac{M_f + M_p + M_L}{M_p + M_L} \tag{8}$$

Define

$$\delta = \frac{M_L}{M_p}. \tag{9}$$

Then, the cost parameter R is

$$R = \frac{M_p + M_b}{M_L} = \frac{\beta + \mu(1+\delta)}{\beta\delta}. \tag{10}$$

Note that βR is just the ratio of the structural weight required for a given mission and payload to the booster structural weight required to put the same payload into a low-altitude earth-satellite orbit.

For an arbitrary exhaust velocity program $v(m)$ we may write

$$V_2 - V_1 = \bar{v}h(\mu) \tag{11}$$

h is determined by the choice for $v(m)$. The treatment of this section is then more general and may include a gravitational field. In the latter case the best choice for the function $v(m)$ would not be Eq. (4). It is convenient to define "propulsion efficiency" by

$$\eta = \frac{(\frac{1}{2})M_2(V_2 - V_1)^2}{E} = \frac{M_2(V_2 - V_1)^2}{M_f\bar{v}^2} \frac{h^2}{\mu - 1}. \tag{12}$$

The justification for this definition is the topic of a separate report.[1] The specific energy k of the propulsion system is

$$k = \frac{E}{M_p}, \tag{13}$$

where E is the maximum energy that can be delivered to the exhaust gases. The maximum value of k is determined by the state of the technology. The parameter

$$\phi = \frac{2k}{(V_2 - V_1)^2}, \qquad \phi - 1 \tag{14}$$

then defines the difficulty of a mission $V_2 - V_1$ in terms of the energy capacity of the power plant. We see that

$$\phi = \frac{M_f\,\bar{v}^2}{M_p(V_2 - V_1)^2} = \frac{M_f}{M_p\,h^2}. \tag{15}$$

Using (9), (10), (13), we then have

$$\phi = \frac{(1+\delta)}{\eta} \tag{16}$$

Substituting into (11), we have

$$R_E(\beta\phi\eta - 1) = \beta + \mu\phi\eta, \tag{17}$$

where R_E is the energy-limited cost index. If we now differentiate with respect to μ with $\phi, \beta =$ constant and set

$$\frac{\partial R_E}{\partial \mu} = 0,$$

we find the minimum value the cost parameter R may

have for a given ϕ, β. The result is

$$(\beta R_E)_{\text{min.}} = \frac{\eta + \mu \dfrac{d\eta}{d\mu}}{\dfrac{d\eta}{d\mu}} \tag{18}$$

From (17), (18), the corresponding expression for ϕ is

$$\phi = \frac{\eta + (\beta + \mu) \dfrac{d\eta}{d\mu}}{\eta^2} \tag{19}$$

Thus, the minimum value R may have for a given ϕ, β is given by the parameteric equations (18), (19).

We can now show that the exhaust velocity program (4) gives the absolute minimum for R_E when β, ϕ are given and there is no gravitational field. Suppose we have a rocket of mass ratio μ, specific energy k, and desire a final velocity V. The energy E is given by

$$E = kM_p\,. \tag{20}$$

So

$$M_L = M_2 - M_p = M_2 - \frac{E}{k}\,. \tag{21}$$

Therefore, if we choose the velocity program which minimizes E we have maximized M_L and, hence, minimized M_p. This makes R_E as small as possible. However, the problem of minimizing E under these conditions is a variational problem described by Eq. (3) with (2) as the auxiliary condition and leads to the same solution (4) as before. For the two cases

$$v(m) = \frac{M_2}{M_1 - m}\sqrt{\mu \bar{v}} \tag{22}$$

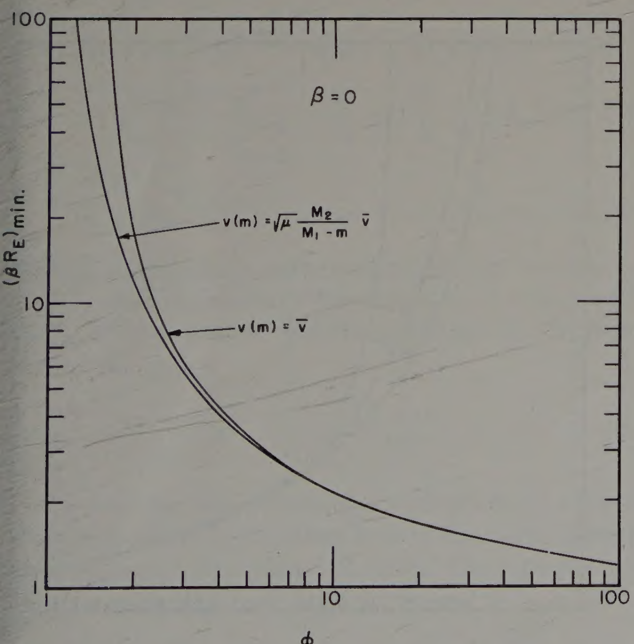

FIG. 2. Variation of minimum payload cost index with mission difficulty for chemical booster ($\beta \ll 1$). Constant and optimized velocity programs are shown. Energy-limited missions.

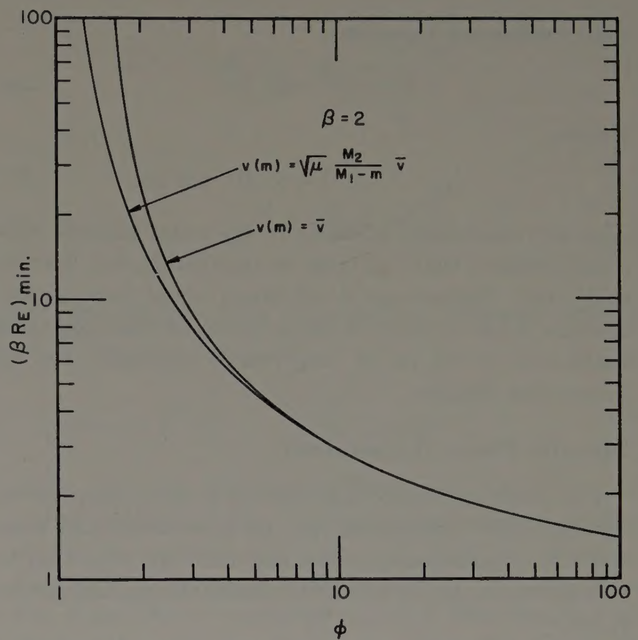

FIG. 3. Variation of minimum payload cost index with mission difficulty for optimistic nuclear hydrogen rocket boosters ($\beta = 2$). Energy-limited missions.

and

$$v(m) = \bar{v}, \tag{23}$$

the efficiency functions η may be computed from (5a), (5b), (12). We have,

$$\eta = \frac{\mu - 1}{\mu} \tag{24}$$

and

$$\eta = \frac{\ln^2\mu}{\mu - 1}, \tag{25}$$

respectively. The parametric equations (18), (19) may now be solved. For a given β, ϕ (booster performance, mission difficulty), (19) gives the value for μ which will minimize R_E. Substitution of this value into (18) then gives the corresponding value for R_E. Figures 2, 3, show the results for $\beta = 0, 2$. It can be shown that for the case v = constant, $\mu < 4.9$. This results from the fact that η has a maximum at $\mu = 4.9$ for this case. Higher values of μ are of course possible but would lead to increased payload cost. From these figures we see that the optimized velocity program gives appreciable benefits in terms of payload cost only for mission requirements which approach the energy limits of the rocket.

Now we can put in some numbers to see what constitutes such an ambitious mission. The most advanced current reactor designs achieve about 4000 to 6000 thermal megawatt days/ton energy release in the core before refabrication or other reprocessing of the fuel elements becomes necessary. Allowing a factor of 2 for the remainder of the structure weight and an overall efficiency of 5% for the propulsion plant (this includes thermodynamic efficiency of the power plant and ion

source efficiency), we have

$$k \simeq 10^{14} \text{ ergs/gm.} \tag{26}$$

Hence,

$$V_2 - V_1 < 1.4 \times 10^7 \text{ cm/sec.} \tag{27}$$

This corresponds to missions to the outer planets with a requirement that the time be reasonable, say 3 years each way. Technological advances could conceivably increase k by a factor of 10 or more. In this case, this limitation would be of importance primarily for an interstellar mission.

Specific Power Limitations

For the 6-year round trip mission to the outer planets, specific power limitations may be more important than the energy limitations, so we now turn our attention to this aspect of the problem. We define the specific power K by

$$K = \frac{P}{M_p}, \tag{28}$$

where P = exhaust power. The first problem is the integration of the equation of motion for arbitrary $v(m)$ subject to the condition $P = KM_p$. We set $P = KM_p$ since, clearly, maximum acceleration will result when the power is maximized. We assume that K is independent of v. If it should turn out that the ion source and accelerator structural weights are small compared to M_p, then this assumption is reasonable. If the rocket thrust is T (a function of the time) then the equation of motion is

$$\frac{dV}{dt} = \frac{T}{M_1 - m}. \tag{29}$$

Now

$$\frac{dm}{dt} = \frac{2p}{v^2(m)} = \frac{2KM_p}{v^2(m)}, \tag{30}$$

so

$$\int_0^m v^2(m)\, dm = 2KM_p t \tag{31}$$

gives $m(t)$ for a prescribed function $v(m)$. Also,

$$T = \frac{2p}{v(m)} = \frac{2KM_p}{v(m)}, \tag{32}$$

which, together with (31), gives $T(t)$. Equation (29) can now be integrated twice to yield $s(\tau)$, the distance travelled during the burning time τ where

$$\tau = \frac{\int_0^{M_f} v^2(m)\, dm}{2KM_p} = \frac{E}{KM_p} \tag{33}$$

Here E is the total energy carried away by the exhaust gases and not the total energy capability as in the previous section. Therefore E is not fixed. The result of the integration has the form

$$s(\tau) = \frac{K}{(1+\delta)} f(\mu)\left(\frac{2E}{M_2}\right)^{3/2}. \tag{34}$$

Similarly,

$$\tau = \frac{E}{KM_p} = \frac{(1+\delta)}{2K}\frac{2E}{M_2} = \frac{(1+\delta)}{2K}(\mu - 1)\bar{v}^2. \tag{35}$$

Combining (34), (35), we have a relation of the form

$$\frac{s^2}{\tau^3} = \frac{K}{(1+\delta)} F(\mu), \tag{36}$$

where $F(\mu)$ depends only upon the choice of the functional form of $v(m)$. Clearly, for missions that are not energy-limited it is desirable to choose μ so that s^2/τ^3 is minimized, since this will give the lowest transit time for a given distance s. If the final velocity is to be small (or zero) rather than arbitrary, then the thrust must be reversed at some time τ_1, such that $V(\tau) = 0$. The form of the result, (36), remains the same, however. For the choice (4) for $v(m)$ it is interesting to note that dV/dt = constant so that $\tau_1 = \tau/2$. In the following, however, we shall not restrict $V(\tau)$.

For a given $v(m)$, the maximization of (36) determines optimum μ and hence the optimum average exhaust velocity $\bar{v}$. Thus, we have a procedure for determining the best value for $\bar{v}$ when δ is fixed. However, the principal problem is the minimization of R, the payload cost index, for a specified mission requirement s^2/τ^3 and specific power capability K. This leads to a variational problem for the determination of $v(m)$ which has not as yet been solved. The results of the previous section do suggest, however, that an interesting class of functions for the exhaust velocity program is

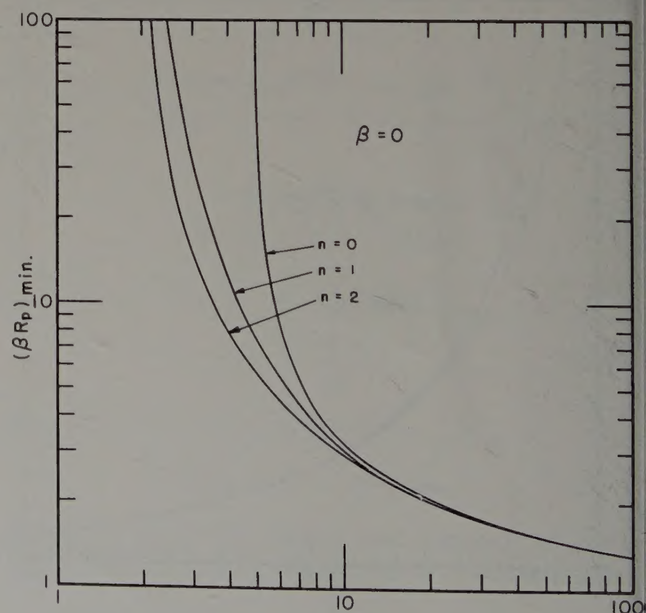

Fig. 4. Variation of minimum payload cost index with mission difficulty and choice of exhaust velocity program for chemical boosters. Power-limited missions.

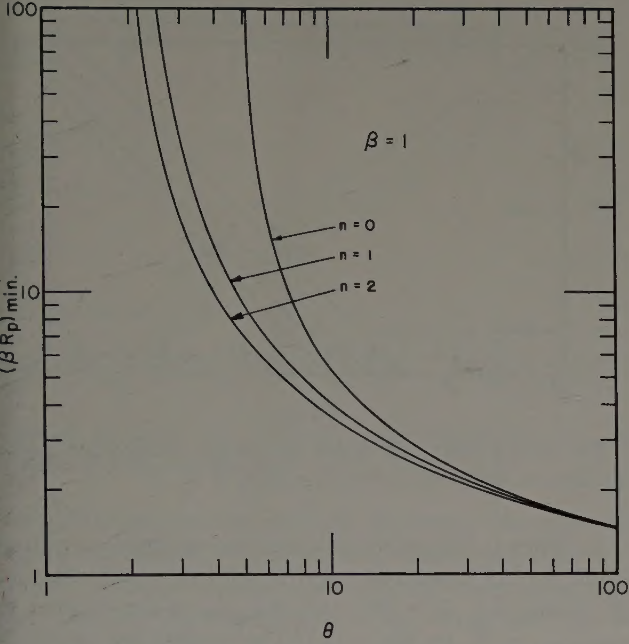

FIG. 5. Variation of minimum payload cost index with mission difficulty and choice of exhaust velocity program for reasonable nuclear hydrogen rocket boosters. Power-limited missions.

$$v(m) = \frac{\nu}{(M_1 - m)^n}. \qquad n > 0 \quad (37)$$

ν can be expressed in terms of τ, μ, n, M_2 by substituting (37) into

$$\frac{1}{2}\int_0^{M_f} v^2(m)\, dm = E = \frac{KM_2\tau}{1 + \delta}. \qquad (38)$$

The resulting form of (36) is

$$\frac{s^2}{\tau^3} = \frac{K}{(1 + \delta)} \frac{2(2n - 1)}{n^2\mu(\mu^{2n-1} - 1)} \cdot \left[\frac{(2n - 1)(\mu^{3n-1} - 1)}{(3n - 1)(\mu^{2n-1} - 1)} - 1\right]^2, \qquad n > 0 \quad (39)$$

$$= \frac{K}{(1 + \delta)} F(\mu, n).$$

We can now define a parameter θ which is analogous to ϕ and which determines the difficulty of a mission in terms of the specific power capability of the power plant. We have

$$\theta = \frac{K\tau^3}{s^2} = \frac{(1 + \delta)}{F}. \qquad (40)$$

We shall see that all missions with the same value for θ are equivalent in that they lead to the same optimum mass ratio and minimum cost index. The expression for the power-limited cost index R_p is now

$$R_p(\beta\theta F - 1) = \beta + \mu\phi F. \qquad (41)$$

Carrying out the minimization process for R as before, we obtain a similar pair of parametric equations

$$(\beta R_p)_{\text{min.}} = \frac{F + \mu \dfrac{dF}{d\mu}}{\dfrac{dF}{d\mu}} \qquad (42)$$

and

$$\theta = \frac{F + (\beta + \mu)\dfrac{dF}{d\mu}}{F^2} \qquad (43)$$

For the simple case of constant exhaust velocity ($n = 0$),

$$F(\mu) = \frac{2[(\mu - 1) - \ln \mu]^2}{(\mu - 1)^3}. \qquad (44)$$

Note that the F-functions given by (39), (44) all approach the limit $(\mu - 1)/2$ as μ approaches 1, as they should. The solutions of Eqs. (42), (43), are shown in Figs. 4, 5, 6 for $n = 0, 1, 2$ and $\beta = 0, 1, 2$. Note from Fig. 6 that for $v =$ constant, $\mu < e$. This results from the fact that (44) has a maximum for $\mu \simeq e$. No results are given for $n > 2$ since no further performance benefits result.

In order to see what these curves mean, let us consider a specific mission,

$$s = 40 \text{ million miles}$$
$$\tau = 65 \text{ days}$$
$$K = 0.1 \text{ kw/kg}.$$

Then

$$\theta = 6.2.$$

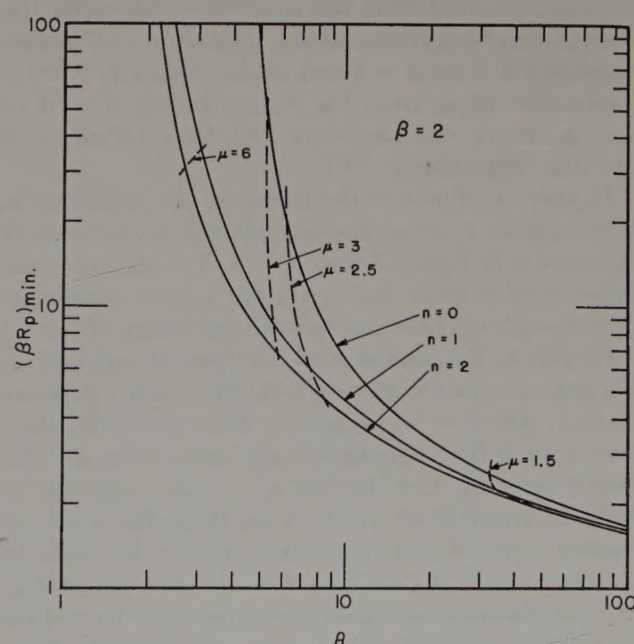

FIG. 6. Variation of minimum payload cost index with mission difficulty and choice of exhaust velocity program for optimistic nuclear hydrogen rocket boosters. Power-limited missions.

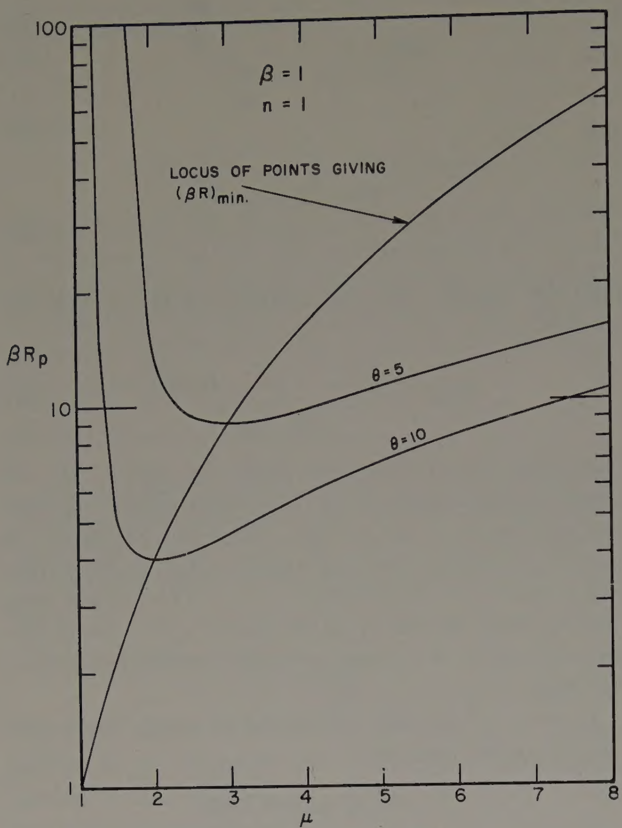

FIG. 7. Nature of the cost index function for a fixed mission θ.

From Fig. 4 we see that there is nearly a factor of 2 in payload cost index between the cases $n = 0, 2$ when $\beta = 0$. For $\beta = 2$ (optimistic nuclear hydrogen rocket booster) the factor is 3. To put the situation differently, consider the effect that the choice of n has upon the mission time for a given cost R_p, distance s, and booster capability β. Take $\beta = 2$ and assume that $\beta R_p = 10$ is reasonable. Then, from Fig. 6, going from $n = 0$ to $n = 2$ reduces τ by the factor 1.86. Even for $\beta R_p = 4$ the time reduction is 30%.

In order to illustrate the nature of the minimum in R_p for a given θ, $\beta R(\mu)$ has been plotted vs μ for various values of θ in Figure 7 with $\beta = 1$, $n = 1$. Note that there exists a lower limit on μ for a given mission θ, even though we have not limited the energy E.

In Fig. 6, the dashed lines are lines of constant μ. We note a rather surprising fact: For "easy" missions ($\theta \gg 1$), the $n = 2$ case gives a lower cost index than does $n = 0$ but requires a *higher* mass ratio μ. This results from the fact that for $n = 2$ the reduction in required power plant weight more than offsets the increased payload and propellant weights. In order to illustrate this point we take $\theta = 6.2$ and compute μ, M_p, M_f for constant exhaust velocity ($n = 0$) and for $n = 1, 2$ from Fig. 6. The booster structural weight is assumed to be unity. If we further assume $\tau = 65$ days, $K = 0.1$ kw/kg, the corresponding value of $\bar{v}$ is also determined.

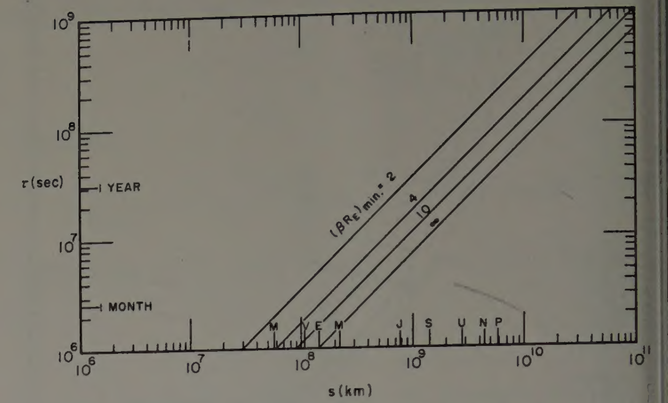

FIG. 8. Payload cost index for energy-limited ion rocket missions in field-free space. $\beta = 2$, $n = 1$.

In order to obtain these numbers, we find μ optimum from (43), F from (44) or (39), δ from (40), M_p and M_f from (6), (7), (8), (9). $\bar{v}_{optimum}$ is determined by $\mu_{optimum}$, τ, K by Eq. (35). The results are shown in Table I. The low values for $\bar{v}$ result from ignoring the gravitational field. They do indicate, however, that for an actual trip from Earth to Mars, the optimum $\bar{v}$ would be close to ΔV_{eff} for such a trip. Note that minimizing the payload cost index led to values for $\bar{v}$ much less than are usually associated with ion rockets.

TABLE I

Effect of Exhaust Velocity Program Upon Ion Rocket Weight Breakdown

	v = constant	$\beta = 2$ $v(m) = (M_1 - m)$	$\theta = 6.2$ $v(m) = (M_1 - m)^2$
M_b	1.0	1.0	1.0
$\mu_{optimum}$	2.5	2.77	2.86
M_p	0.646	0.365	0.287
M_f	1.20	1.28	1.30
M_L	0.154	0.355	0.413
$\bar{v}_{optimum}$, (cm/sec)	8.64×10^5	7.96×10^5	7.76×10^5
$(\beta R_p)_{min.}$	20.5	7.5	6.3

Concluding Remarks

The results of the preceding sections can be illustrated graphically by plotting curves of constant $(\beta R)_{min.}$ resulting from both the specific power and specific energy considerations in the s, τ plane. This is done for the velocity program $n = 1$, $\beta = 2$, and $k = 10^{14}$ erg/gram, $K = 0.10$ watt/gram in Figs. 8 and 9, respectively. Mean distances of the various planets from the sun are indicated on the axis of abscissas.

Now, the energy limitation exists only for a mission time $\tau > \tau_b = k/K$, where τ_b is the "burnout" time for the propulsion plant. For $t > \tau_b$ the rocket would coast. This situation is shown in Fig. 10. Since for $n = 1$ the acceleration is constant,

$$V = \frac{\bar{V}}{2} = \frac{s_b}{2\tau_b}.$$

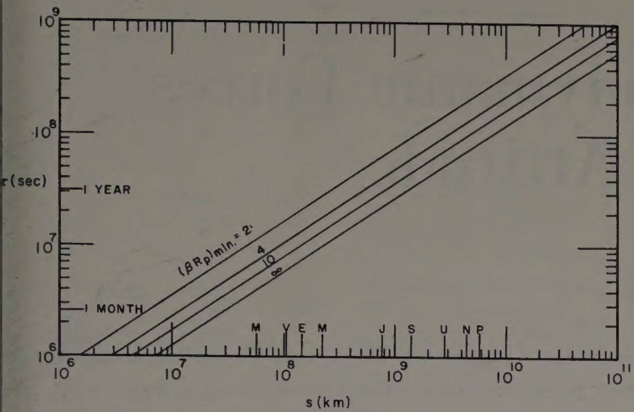

FIG. 9. Payload cost index for power-limited ion rocket missions in field-free space. $\beta = 2$, $n = 1$.

The curved part of the $(\beta R)_{\text{min.}}$ lines show the coasting part of the flight. It should be pointed out that the straight sections of these lines are *not* trajectories, but rather the loci of the points s, τ for which $(\beta R)_{\text{min.}}$ is constant. Since $n = 1$ optimizes R for $\tau > \tau_b$ while $n = 2$ is nearly optimum for $\tau < \tau_b$, the curves shown here do not give optimum results. However, they do show clearly the general nature of the results to be expected when an optimization is carried out.

The picture presented here undergoes major changes if real gravitational fields are considered. For example, $V_2 - V_1$ is replaced by the characteristic velocity of the mission V_c. V_c is defined by

$$V_c = \int_0^\tau a(t)\, dt$$

$a(t)$ = acceleration due to thrust forces alone.

For values of the acceleration which appear reasonable for ion propulsion systems (1 milli-g $> a >$ 0.05 milli-g) V_c is independent of a if minimum energy transfer orbits are used. In this case, it turns out that the equations for energy-limited flight are applicable. If, instead, we disregard energy consumption but try to minimize transit time, an entirely different sort of exhaust velocity program is used. Also, minimizing one-way transit time is a different problem than minimizing round trip time because of the influence of the motions of the planets. In general, then, the situation is considerably more complicated when actual gravitational fields are included. It remains true, however, that appreciable gains in performance can be achieved by proper programming of the exhaust velocity.

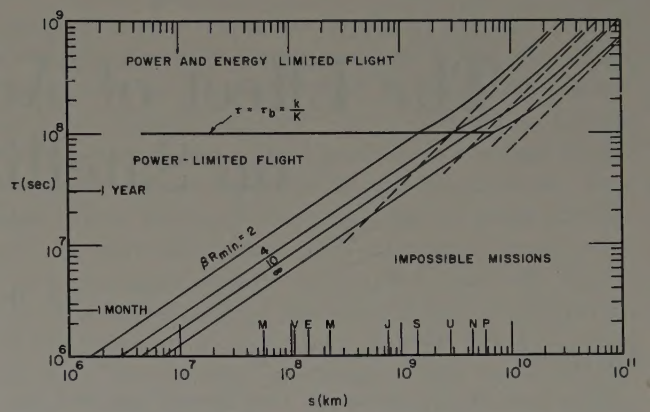

FIG. 10. Payload cost index for ion rockets in field-free space

References

1. R. H. Fox, "Optimum Exhaust Velocity Programming and Propulsion Efficiency", University of California Radiation Laboratory Report No. UCRL-5135, February 12, 1958 and *Journal of the Astronautical Sciences* [American Astronautical Society]. Vol. VI No. 1.

The Effect of Aerodynamic Forces on Satellite Attitude

D. B. DeBra*

Abstract

The effects of torques due to aerodynamic drag and the gravity gradient are computed for satellites orbiting at altitudes of between 80 and 375 miles. Motion about the pitch axis is discussed and the equilibrium position is determined as a function of altitude for both a dumbbell and cylindrical-shaped object. The equilibrium position of the cylindrical object in three dimensions is considered as a function of altitude. Equations are presented in the appendix.

Introduction

It is often desirable to place in orbit a satellite which is oriented in a unique attitude with respect to the earth. If a stable attitude can be found, attitude control is reduced to a damping problem. Two methods have been proposed to obtain such a stable attitude. These methods use, (1) the gradient in the earth's gravitational field, and (2) the aerodynamic torques. Each method requires that the vehicle possess certain properties. In gravity gradient stabilization the vehicle must have a single axis about which the moment of inertia is at a minimum. In the stabilized position this axis thus becomes aligned to the vertical [2, 6, 7]. In aerodynamic stabilization it is necessary for the center of pressure to lie behind the center of mass in the direction of vehicle velocity.

Discussion

Gravity gradient stabilization can be explained best if we visualize a dumbbell shaped object in the earth's field. In a horizontal position the forces on each mass (ball) are the same, but in other than this exact position there is more attraction on the mass closer to the earth. Therefore, in addition to the net force there is a moment acting unless the dumbbell is aligned to a vertical position.

It is difficult to design a vehicle without consideration of both gravity gradient and aerodynamic torques. When attitude stabilization is attempted utilizing either effect, disturbances may be introduced from the other. The problem considered here is a study of the satellite stable equilibrium position in a circular orbit, under the influence of a gravitational field, moving in a model atmosphere.

Two nominal vehicles have been chosen to illustrate this problem and are shown in Fig. 1 and 7. The geometries are distinct but for simplification the physical properties chosen are the same. The analysis proceeds from a consideration of behavior about a single axis, the pitch axis (discussed in ref. 1), to a three dimensional study of the cylinder.

Aerodynamic forces at altitudes above 80 statute miles make an interesting study in themselves. The computation of drag based on free molecular flow is complex [3]. The most exact theory takes into account not only the random velocity of the particles as they approach the vehicle but also as they leave the surface after impact. The molecular speed ratio is the ratio of vehicle velocity to the mean particle velocity. When this ratio is high, the effect of the incoming particles can be computed quite accurately by ignoring the random components. When random motion is only considered as the particles leave the vehicle surface, the theory is described as Newtonian-diffuse.

Particle random velocity defines the ambient temperature. At these altitudes the temperature may be around 1500°R. After a particle strikes a surface it loses some of its energy and leaves with a velocity associated with a temperature near the surface temperature of the vehicle. This may be around 550°R. It would seem natural, then, to neglect the momentum change of particles leaving the surface since their mean velocity is even smaller than the random velocity ignored in the Newtonian-diffuse analysis. But, in some cases, this is not true since the particles' effect is assumed normal to the vehicle surface. The relative force may be small but the resultant moment can be significant. When the effect of particles leaving the surface is also ignored the flow is Newtonian and the computation is considerably simplified.

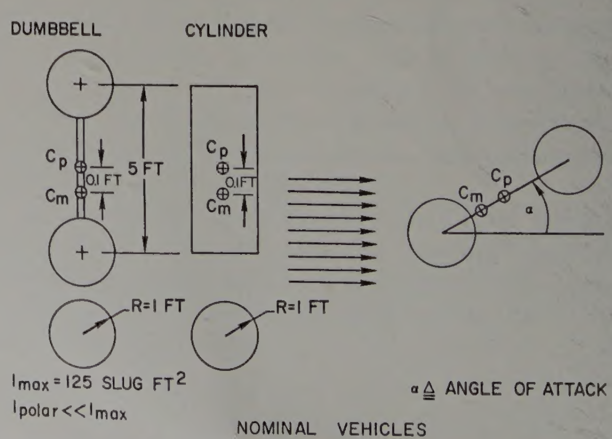

FIGURE 1

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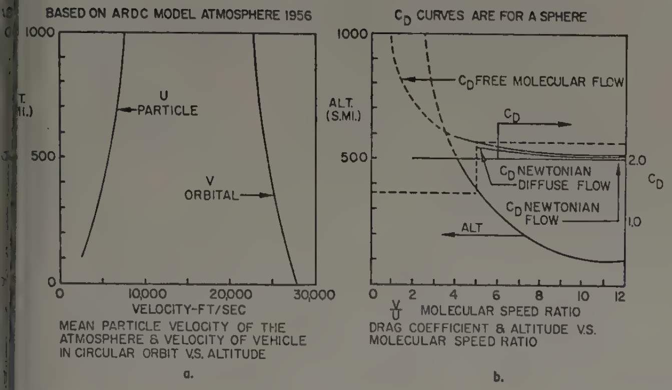


FIGURE 2

Newtonian flow possesses a drag coefficient of two. Thus, when the drag coefficient based on free molecular flow approximates two, the assumption of Newtonian flow yields reasonable results. Figure 2 presents a method of determining in what range of altitude this approximation can be made. The drag characteristic plotted is for a sphere but the shape of the curve is characteristic. The assumption of Newtonian flow is therefore justifiable for altitudes less than 375 statute miles for accuracies near 10 percent.

The Knudsen number is the ratio of particle mean-free-path to a significant vehicle dimension. From experimental results flow is essentially free-molecular for Knudsen numbers greater than two. For the vehicles under consideration here, the Knudsen number is approximately five at an altitude of 80 statute miles and increases rapidly above that.

All computations in this paper are based on Newtonian flow and are a good approximation for altitudes between 80 to 375 statute miles.

First let us consider the vehicles of Fig. 1. With freedom of motion only about an axis normal to the plane of the orbit, the variable describing the attitude is merely the angle α defined in Fig. 1.

Consider a vehicle stabilized by the gravity gradient so that its position corresponds to $\alpha = 90$ degrees. For the assumed separation of the center of mass and the center of pressure of 0.1 ft, the torque required of a control system is shown in Fig. 3. This assumes that a

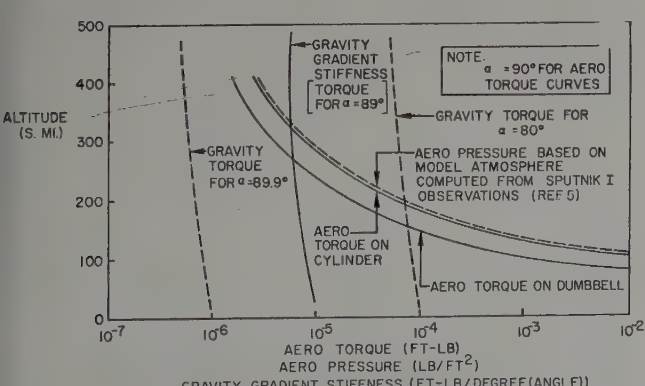


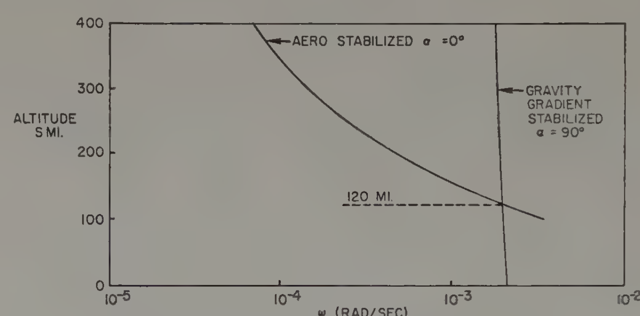
FIGURE 3

constant torque must be produced by the control system in order to maintain position. If not, the vehicle will rotate toward the position in which the gravity gradient torque balances the aerodynamic torque. Since a control system that produces a constant torque is expensive in weight and power, the attitude error that can be tolerated determines the minimum altitude of a circular orbit when constant control is not used.

For the model considered, it may be correctly argued that the deviation from the vertical can be planned on and should not be considered an error. For a satellite, however, the aerodynamic forces would not be as constant as they are in the model. Normal fluctuations in the atmosphere and variations in the vehicle altitude cause a variation in the equilibrium position. Since the variations may be close to the deviation from the vertical, where α nears 90° the deviation is treated as an error.

The aerodynamic pressure decreases with altitude but theoretically never vanishes. Therefore, if there is an area moment about the center of mass that can produce an aerodynamic torque, there is no altitude at which the equilibrium attitude of the vehicle exactly coincides with the vertical. However, the error becomes insignificant for higher altitudes.

When the vehicle is stabilized by aerodynamic effects ($\alpha = 0$) at a given altitude, the restraining force is a function of the deviation. The frequency of oscillation about the equilibrium position is a convenient method of comparing the stiffness of constraint for small deviations.



ALTITUDE VS NATURAL FREQUENCY OF VEHICLE FOR GRAVITY GRADIENT ACTING ALONE AND AERODYNAMIC FORCES ACTING ALONE

FIGURE 4

Figure 4 presents the natural frequencies of these vehicles for small oscillation about equilibrium positions determined in the absence of all torques except the constraining torque indicated.

It is characteristic of the gravity gradient torque to vary approximately as the sine (2α). In other words, there is no constant disturbing torque due to the gravity gradient when one of the vehicles is in a horizontal position ($\alpha = 0$ degree or 180 degrees). This does not mean that the position is stable. When α is near 180 degrees, the slightest disturbance allows both the aerodynamic and gravity torques to increase the deviation. This condition is clearly unstable but the position

near $\alpha = 0$ degree is not. At α near 0 degree, the restraining torque due to aerodynamic forces may be less for small deviations than the gravity gradient divergent torques.

The crossover point (where the aerodynamic restraining torques equal the gravity gradient disturbing torques) is an altitude below which the position, $\alpha = 0$ degree, is a stable equilibrium position for aerodynamic stabilization. By the nature of the gravity gradient torques the effect near $\alpha = 0$ degree are equal in magnitude to those near $\alpha = 90$ degrees, but reversed in sign. Therefore, Fig. 4, which displays the natural frequencies, also indicates the altitude at which this crossover occurs.

For the vehicles chosen in this part of the study, aerodynamic stabilization can be accomplished without theoretical error for orbits below 120 statute miles.

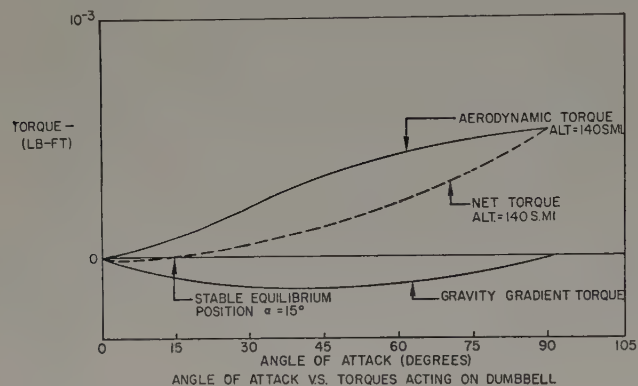


FIGURE 5

The two important considerations for a stabilized vehicle are its equilibrium position and how stiffly it is held in this position. A method of displaying both is shown in Fig. 5, where the net torque acting on the vehicle is plotted as a function of vehicle attitude for a certain altitude. The stable equilibrium position is the point where the net torque passes through zero with positive slope, and the slope indicates the stiffness of constraint.

Figure 6 shows the equilibrium position as a function of altitude. The rapid change in position with altitudes near 150 statute miles should be noted. Variation in the atmosphere and in vehicle altitude would have the greatest effect on the equilibrium position at this altitude.

An additional source of torque must be considered when the three dimensional case is analysed. It is a gyroscopic or precessional torque. The reference frame used in the previous analysis involves the vertical which rotates in space as the satellite circles the earth. A body which is fixed with respect to that reference has the same angular velocity. Components of the angular velocity along the principal axes define angular momenta, which, in general, are changing direction. For a

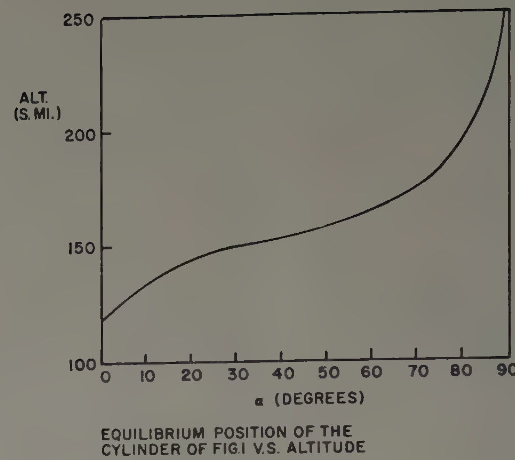


FIGURE 6

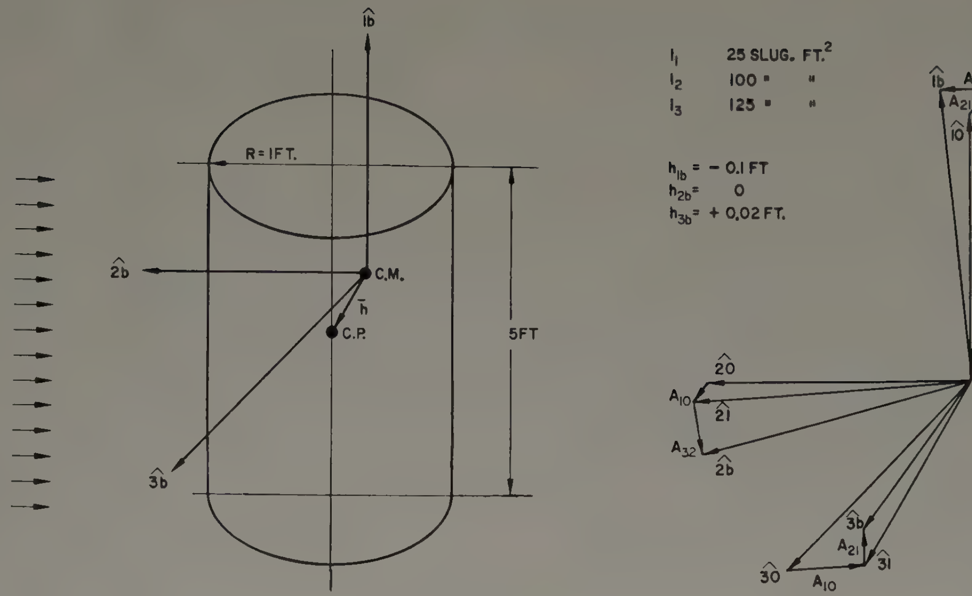
given position, this results in torques about the body axes and a coupling of the motion between the axes.

One effect is to make the equilibrium position of the satellite unique when the moments of inertia about the principal axes are distinct. This can be verified by a simple experiment. If an object with three distinct principal moments of inertia, such as a book, is spun about a principal axis as it is thrown into the air, it is found that the motion is stable about the axis of maximum or minimum moment of inertia, but unstable about the axis of the intermediate moment of inertia. Apply this to a satellite in the absence of aerodynamic torques when the satellite axis of minimum moment of inertia is constrained to the vertical by the gravity gradient torque. Then, to attain attitude stability, the axis of maximum moment of inertia must be aligned with the orbital angular velocity vector.

To study the effect of aerodynamic torques when there are three degrees of angular freedom for the cylinder, consider the previous analysis to be altered by changes in the cylinder, as shown in Fig. 7. The change in the moments of inertia reduces the gravity gradient torques by a factor of nearly two and the location change of the center of pressure produces aero torques about the other axes. The resultant equations, which are highly non-linear, are developed in the appendix (10).

There are many ways of defining a vehicle attitude. The method used here is common and involves the angles as defined in Fig. 7. The effects of the atmospheric torques are described by the behavior of these angles as a function of altitude, as shown in Fig. 8.

There are two results of primary interest. One occurs when the aerodynamic effects have tipped the vehicle over to a horizontal position and the axis of minimum moment of inertia is no longer aligned to the vertical. The satellite does the "next best thing" and aligns the axis of intermediate moment of inertia to the vertical. The second is that for small deviations from the aero torque-free position, the equilibrium position can be computed roughly by using the spring stiffness given for each axis (11).



CYLINDER & COORDINATES FOR 3D ANALYSIS

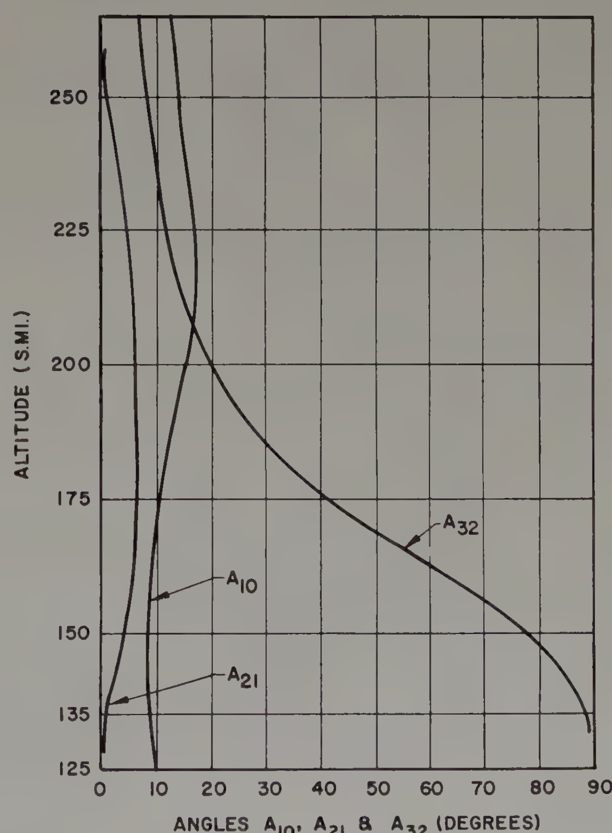
FIGURE 7

Conclusions

It should be emphasized that the results presented here apply to certain vehicles and differ greatly as the vehicle characteristics change. For instance, the ratio of gravity-gradient torques to aerodynamic torques varies as the square of the size for similar vehicles, and greater effects can be expected if specific properties of a vehicle are changed.

If a vehicle is designed specifically for gravity gradient stabilization, it is likely that a center-of-mass–center-of-pressure separation could be controlled to less than two percent of the vehicle length. Similarly, aerodynamic stabilization could be achieved more easily in a vehicle which is designed with nearly constant moments of inertia about all axes.

Although any position of the satellite vehicle may be maintained by the use of an attitude control system, the power requirements for such a system could be prohibitively high in any satellite designed for long orbital life. For vehicles designed for a stable equilibrium position—so that the potentially complex control function is reduced to a damping problem—allowances must be made for the effects of extraneous torques which could change the equilibrium position beyond the design tolerances of the vehicle attitude.



ATTITUDE OF CYLINDER OF FIG 7 AS A FUNCTION OF ALTITUDE

FIGURE 8

References

[1] D. B. DeBra and E. V. Stearns, "Problems of Attitude Control of Satellite and Interplanetary Vehicles," *AIEE* No. 58-961, 24 Jun 1958.

[2] William R. Davis, "Determination of a Unique Attitude for an Earth Satellite," *Proceedings AAS*, Jan 1958 No. 57-10, Revised 15 Nov 1957.

[3] W. A. GUSTAFSON, "The Newtonian-Diffuse Method for Computing Aerodynamic Forces and Moments on Bodies at High Altitudes," *Lockheed Report*, 6 Jun 1958.
[4] "ATTITUDE CONTROL": A Panel Discussion, *Navigation*, Vol. 6, No. 1, Spring 1958.
[5] JOHN BREAKWELL AND LEIGHTON KOEHLER, "Elliptic Orbit Lifetimes," *AAS 58-34*.
[6] ROBERT E. ROBERSON, "A Review of the Current Status of Satellite Attitude Control," presented at *IAS-ARS Meeting*, San Diego, Calif., 5 Aug 1958.
[7] R. E. ROBERSON AND D. TATISTCHEFF, "The Potential Energy of a Small Rigid Body in the Gravitational Field of an Oblate Spheroid," *J. Franklin Inst.*, No. 262, 1956, pp. 209–219.
[8] BRIAN F. DOOLIN, "The Application of Matrix Methods to Coordinate Transformations Occurring in Systems Studies Involving Large Motions of Aircraft," *NACA TN3968*, May 1957.
[9] H. GOLDSTEIN, "Classical Mechanics," *Addison-Wesley Publishing Co.*, pp. 133, 147, 158.
[10] F. B. HILDEBRAND, "Advanced Calculus for Engineers," *Prentice-Hall, Inc.*, p. 353.

Appendix

Nomenclature

$\bar{L}$	Angular momentum of satellite
$\bar{N}$	Torque acting on satellite
$\bar{W}$	Angular velocity of satellite
$\bar{h}$	Displacement of center of pressure re. the center of mass
$\bar{F}$	Force
α	Angle of attack of the body axis of symmetry
$[A]$	Matrix defining transformation from orbit coordinates to body coordinates
a_{ij}	Elements of $[A]$
$[A_{ij}]$	Matrix of transformation of jth triad into $(j+1)$st triad through an angle A_{ij} about the ith component of the jth coordinate triad.
C_{ij}	Cosine of angle A_{ij}
S_{ij}	Sine of angle A_{ij}
I_{ijk}	The ijth elements of the intertia tensor expressed in the kth coordinate triad
I_i	The moment of inertia about the ith principal axis of the body.
–	Denotes a vector
$\widehat{ij}$	A unit vector along the ith component of the jth coordinate triad.

The coordinate triads are defined by the subscripts 0, 1, 2, b which define coordinate triads successively from the orbit triad after transformation $[A_{10}]$, $[A_{21}]$ and $[A_{32}]$ as shown in Fig. 7. The final coordinate triad, b, corresponds to the principal body axes in which the magnitude of the principal moments of inertia, I_1, I_2, and I_3 $(I, < I_2, < I_3)$.

The components of triads are numbered successively 1, 2, 3 and form a right-handed coordinate system.

Subscripts a and g refer to aerodynamic and gravitational effects.

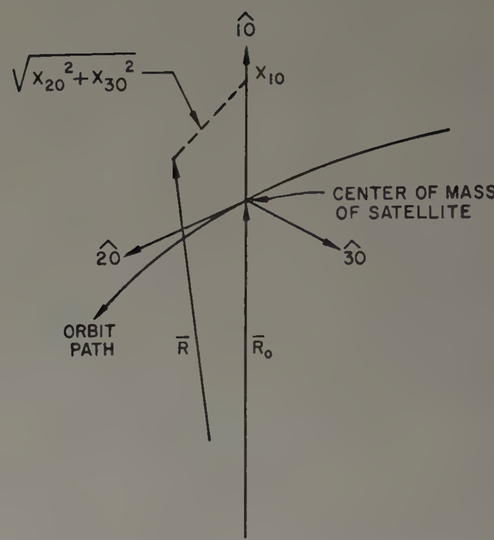


FIGURE 9

Equations of the Applied Torques

The basic equation for angular motion is

$$\frac{d}{dt}\bar{L} = \bar{N} \tag{1}$$

where

$$\left(\frac{d}{dt}\right)_{\text{fixed}} = \left(\frac{d}{dt}\right)_{\text{rotating}} + \bar{W} \times \quad \text{(Ref. 9)} \tag{2}$$

Since the equilibrium positions are sought the transformation $[A]$ is constant and

$$\left(\frac{d}{dt}\right)_{\text{rotating}} = 0 \tag{3}$$

in both body and orbit coordinate frames. Computation in principal body axis results in considerable simplification because the inertia tensor is diagonalized and moments of inertia are constant. The two sources of torque considered are (1) Aero dynamic torque and (2) gravity gradient torque acting on the cylinder of Fig. 7.

(1) *Aerodynamic Torque (For Newtonian diffuse method*, see [3])

Assuming Newtonian flow, a center of pressure and therefore an $\bar{h}$ exists for the vehicle of Fig. 7. Then

$$\bar{F}_0 = F_{20}\widehat{20} = -pS\widehat{20} \tag{4}$$

and

$$\bar{F}_b = [A]\begin{bmatrix} 0 \\ F_{20} \\ 0 \end{bmatrix} = -pS\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}. \tag{5}$$

Since $\cos\alpha = a_{12}$, and $S = \pi\cos\alpha + 10\sin\alpha$, the

rojected area can be found and p, the aerodynamic ressure, is determined from Fig. 3. Therefore

$$(\bar{N}_a)_b = (\bar{h} \times \bar{F})_b = pS \begin{bmatrix} h_3 a_{22} \\ h_1 a_{32} - h_3 a_{12} \\ -h_1 a_{22} \end{bmatrix} \tag{6}$$

(2) *Gravity Gradient Torque*

This torque is computed from the potential energy ne body possesses as a result of its attitude, i.e., the otential can be divided into a potential of the total nass if located at the center-of-mass plus a potential of he body about its center. A spherical mother body is ssumed here but the method is the same when oblate- ess is included. This has been done in [7] and is briefly utlined below.

To find the total potential for a rigid body, the unit otential can be expanded in a Taylor series and in- egrated over the body. The significant term of the otential using $W_0^2 = g_0/R_0$ and coordinates of Fig. 9 is

$$-(I_{220} + I_{330} - 2I_{110})W_0^2/2$$

The introduction of the moments of inertia allows a onvenient method of expressing this part of the po- ential as a function of the angles of a coordinate trans- ormation. [8 and 9]

$$[I]_0 = [A]^{-1}[I]_b[A]$$

The gradient of a potential is a torque when the co- rdinates are angles. The component torques are about he axis for which the angle rotation is defined.

It is appropriate here to discuss the choice of co- rdinates to define a satellite attitude, that is, to choose he form of the transformation $[A]$. Two types are ommon: (1) Classical Euler angles are defined by three otations, the last of which is about the same coordi- ate axis as the first [9]. There are many variations in he choice of first and second axis, but all have the bove property; (2) The only distinct procedure for a hree-angle transformation is to have one rotation bout each of the coordinate axes [8].

For either type of transformation it is most con- enient in this application to choose the first rotation bout the axis of symmetry of the potential function, he vertical. Because of the symmetry, the potential unction will not be a function of rotations about this xis; therefore the labor in finding the torques is re- uced.

When small motions are instructive in understanding he physics it is easier to use the second transformation.

Therefore defining

$$[A] = [A_{32}][A_{21}][A_{10}]$$

the torque in terms of the principal moments of inertia is

$$(\bar{N}_g)_b = 3W_0^2 \begin{bmatrix} S_{21}C_{21}S_{32}(I_1C_{32}^2 + I_2S_{32}^2 - I_3) \\ S_{21}C_{21}C_{32}(I_1C_{32}^2 + I_2S_{32}^2 - I_3) \\ S_{32}C_{32}C_{21}^2(I_1 - I_2) \end{bmatrix} \tag{7}$$

Equations of the Equilibrium Position

Substituting (2) and (3) into (1)

$$\bar{W} \times \bar{L} = \bar{N}_a + \bar{N}_g \tag{8}$$

letting $L_{1b} = I_1 W_{1b}$ etc. and

$$\bar{W}_b = [A]\bar{W}_0 = W_0 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \tag{9}$$

then (8) becomes

$$\begin{aligned} W_{2b}W_{3b}(I_2 - I_3) + (N_a)_{1b} + (N_g)_{1b} &= 0 \\ W_{3b}W_{1b}(I_3 - I_1) + (N_a)_{2b} + (N_g)_{2b} &= 0 \\ W_{1b}W_{2b}(I_1 - I_2) + (N_a)_{3b} + (N_g)_{3b} &= 0 \end{aligned} \tag{10}$$

where the components of $\bar{N}_a$, $\bar{N}_g$ and $\bar{W}$ are given in (6), (7), and (9) as functions of A_{10}, A_{21}, and A_{32}.

Small Angle Expressions

The physical cross-coupling due to the angular velocities relative to the orbit coordinates vanishes for an equilibrium position when the vehicle is at rest relative to the orbit coordinates. The coupling with the orbit rate (due to the rotation of the orbit coordinates) still exists. This coupling combines with the gravity gradient to form an effective spring restraint from which small error angles due to external torques can be estimated. For the aerodynamic torques the error angles are approximately:

$$\begin{aligned} A_{10} &\approx \frac{(N_a)_{1b}}{(I_3 - I_2)W_0^2} \\ A_{21} &\approx \frac{(N_a)_{2b}}{4(I_3 - I_1)W_0^2} \\ A_{32} &\approx \frac{(N_a)_{3b}}{3(I_2 - I_1)W_0^2} \end{aligned} \tag{11}$$

First Order Error Propagation in a Stagewise Smoothing Procedure for Satellite Observations

Peter Swerling*

Abstract

A practical method of smoothing satellite data by evaluating a finite number of parameters, or elements, is presented.

Introduction

The subject discussed is that of smoothing observational data in cases where the observations, in the absence of observational error, would all be determined by the time of observation plus a finite number of parameters, called elements. The objective is to estimate the elements.

The immediate motivation for this arises in studies of estimation of earth-satellite orbits from observational data. In this case, if, for example, the force field were known exactly, one could regard the elements as the position and velocity components at a particular instant t_0; if the field were a central inverse square field, the elements could alternatively be the six elements of a Keplerian elliptic orbit. (It should be mentioned, however, that satellite orbit prediction is only one of the possible applications of the results.)

In satellite tracking and prediction, it is desired to produce ephemerides—i.e., predictions of the future position as a function of time—as well as to make various other types of decisions and predictions. As new observations become available, one can improve the accuracy of these predictions. Ideally, one would like to store all previous observations of the object, and combine these in some optimum way to yield the desired predictions or decisions. The optimum method for processing the available data would be based on analysis of the error statistics for the individual observations, and on the functional dependence of future predicted quantities on the previous observations.

In satellite tracking one is dealing with a situation in which there may be a large number of observations of varying degrees of accuracy, as well as large numbers of tracked objects. Also, methods of orbit prediction (even in the absence of observation errors) are subject to various sources of error, such as

(a) uncertainty in the forces acting on the body (earth's gravitational and magnetic fields, atmospheric resistance);

(b) cumulative errors in solving the equations of motion.

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Two features would be desirable in a tracking and prediction method:

(1) The data processing load per tracked object should not exceed a certain maximum, regardless of how many observations are available to be processed. On the other hand, the prediction method should not throw available observations away.

(2) The method should be adaptable to situations in which the underlying prediction functions are subject to the above-mentioned uncertainties.

The stagewise procedures described below are motivated by these considerations.

The particular methods of data smoothing to be discussed are variations of the classical method of minimizing a quadratic form in the residuals (in practise this is usually a weighted sum of squares of residuals). After defining this method, the first order dependence of the errors in the resulting element estimates on the observation errors is established.

We then go on to describe a stagewise procedure for processing the observational data, in which the element estimates at each stage are smoothed in a particular way with some additional observations. This is, in essence, a type of differential correction procedure. It is shown that the errors in the resulting element estimates, after stagewise smoothing of a given set of observations, have the same first order dependence on the observation errors as would the errors in the estimates obtained by simultaneous processing of the same total set of observations.

Some statistical properties of errors in the element estimates are derived for the case in which the observation errors are regarded as statistical variables and in which the matrix of the quadratic form to be minimized is the inverse covariance matrix of the observation errors. (For Gaussian error statistics, this would result in a maximum-likelihood method of estimation.)

The elements are at first regarded as constants, and later the treatment is extended to the case in which the elements are regarded as time-dependent.

We assume there are N observed quantities F^μ, $\mu = 1, \cdots, N$, each F^μ being a real scalar. We also assume that n real constants x_i, $i = 1, \cdots, n$ exist such that in the absence of observation error, all N observed quantities would satisfy the relations

$$F^\mu = f^\mu(x_1, \cdots, x_n, t_\mu) \tag{1}$$

The vector $x = (x_1, \cdots, x_n)$ will be called the "element vector," and its components the "elements"; t_μ is the time at which the μ^{th} observation is taken.

When observation errors are present, the F^μ are given by

$$F^\mu = f^\mu(x_1, \cdots, x_n, t_\mu) + \mathcal{E}^\mu. \tag{2}$$

For purposes of illustration, consider the observation of a satellite following a Keplerian orbit. The elements might then be taken to be the eccentricity, semi-major axis, inclination, etc.; the quantities F^μ might be observations of such things as range, azimuth, elevation, or range rate from particular observation sites; the f^μ would be the functions describing the dependence of these observed quantities on the elements and time; and the $\mathcal{E}^\mu$ would be the observation errors.

The times t_μ need not be in any particular temporal order, nor are they necessarily all distinct. They are assumed to be measured without error in all cases in which the functions f^μ depend explicitly on t_μ. There is no loss of generality in this assumption, since timing errors may always be reduced to equivalent observation errors; this would of course modify the statistics of the resulting equivalent observation errors (for example, this might introduce an additional systematic component into the equivalent observation errors).

Many of the formulas below will involve the functions f^μ and their partial derivatives $\partial f^\mu/\partial x_i$. Practical application of these formulas would be possible both for cases where analytic expressions are known for the f^μ and their partial derivatives and for cases where these functions must be evaluated by numerical integration, as well as for cases where some of the functions have known analytic expressions and others must be determined by numerical methods.

Henceforth it will be supposed that $N \geqq n$. The problem to be considered is the estimation of the elements by means of smoothing, in some sense, of the observations.

A classical smoothing method is as follows: writing $P = (P_1, \cdots, P_n)$ for an estimate of the element vector, and writing $f(x, t)$ for $f(x_1, \cdots, x_n, t)$, $f(P, t)$ for $f(P_1, \cdots, P_n, t)$, and so forth, the method consists of minimizing with respect to $P_1, \cdots, P_n$ the quadratic form

$$Q = \sum_{\mu,\nu=1}^{N} \eta_{\mu\nu}[F^\mu - f^\mu(P, t_\mu)][F^\nu - f^\nu(P, t_\nu)] \tag{3}$$

where $(\eta_{\mu\nu})$ is a symmetric, positive definite matrix. Thus, the method consists in minimizing a positive definite quadratic form in the residuals. Differentiating Q with respect to P_i and setting the results equal to zero, we find that the minimizing estimates P_i must satisfy

$$\sum_{\mu,\nu=1}^{N} \frac{\partial f^\nu}{\partial x_i}(P, t_\nu)[F^\mu - f^\mu(P, t_\mu)]\eta_{\mu\nu} = 0 \qquad (i = 1, \cdots, n) \tag{4}$$

It is clear that if $\mathcal{E}^\mu = 0$, all μ, then $P = x$ is a solution of (4). The first question to be investigated is that of the first order propagation of errors—i.e., the first order dependence of $P - x$ on the errors $\mathcal{E}^\mu$. It will be assumed that the functions f^μ are sufficiently well behaved for the following operations to be valid. We may write

$$f^\mu(P, t_\mu) = f^\mu(x, t_\mu) + \sum_{j=1}^{n} \frac{\partial f^\mu}{\partial x_j}(x, t_\mu)(P_j - x_j) + \cdots \tag{5}$$

$$\frac{\partial f^\mu}{\partial x_i}(P, t_\mu) = \frac{\partial f^\mu}{\partial x_i}(x, t_\mu) + \sum_{j=1}^{n} \frac{\partial^2 f^\mu}{\partial x_i\, \partial x_j}(x, t_\mu)(P_j - x_j) + \cdots \tag{6}$$

Neglecting all terms of higher order in $P - x$, and substituting (5) and (6) into (4), we obtain

$$\sum_{\mu,\nu=1}^{N} \eta_{\mu\nu}\left[\frac{\partial f^\nu}{\partial x_i}(x, t_\nu) + \sum_{j=1}^{n} \frac{\partial^2 f^\nu}{\partial x_i\, \partial x_j}(x, t_\nu)(P_j - x_j)\right] X\left[F^\mu - f^\mu(x, t_\mu) - \sum_{j=1}^{n} \frac{\partial f^\mu}{\partial x_j}(x, t_\mu)(P_j - x_j)\right] = 0 \tag{7}$$

It is also clear that for sufficiently small $\mathcal{E}^\mu$ and $P_j - x_j$, the term involving second derivatives of f^ν may be neglected. The result may conveniently be expressed as follows: let

$$a_i^\mu(x, t_\mu) = \frac{\partial f^\mu}{\partial x_i}(x, t_\mu) \tag{8}$$

$$\rho_i(x) = \sum_{\mu,\nu=1}^{N} \eta_{\mu\nu} a_i^\nu(x, t_\nu)\,[F^\mu - f^\mu(x, t_\mu)] \tag{9}$$

$$\rho(x) = \{\rho_1(x), \cdots, \rho_n(x)\} \tag{10}$$

$$B_{ij}(x) = \sum_{\mu,\nu=1}^{N} \eta_{\mu\nu} a_i^\nu(x, t_\nu) a_j^\mu(x, t_\mu) \tag{11}$$

$$B(x) = \{B_{ij}(x)\} \tag{12}$$

Then, assuming $B(x)$ to be non-singular,

$$P - x = [B(x)]^{-1}\rho(x) \tag{13}$$

Eq. (13) expresses the first order dependence of $P - x$ on the errors $\mathcal{E}^\mu$. This can be expressed equivalently:

$$P_i - x_i = \sum_{\mu=1}^{N} \Gamma_i^\mu(x)\,[F^\mu - f^\mu(x, t_\mu)] \tag{14}$$

$$\Gamma_i^\mu(x) = \sum_{j=1}^{n}\sum_{\nu=1}^{N} [B(x)]_{ij}^{-1} \eta_{\mu\nu} a_j^\nu(x, t_\nu) \tag{15}$$

A special case of this is as follows: suppose one has already an estimated element vector $p = (p_1, \cdots, p_n)$ together with K new observations. One can form a new estimate vector P by smoothing the original estimate vector p with the new observations in the following manner: P is determined by minimizing, with respect to $P_1, \cdots, P_n$, the quadratic form

$$Q = \sum_{\mu,\nu=1}^{n} \eta_{\mu\nu}(p_\mu - P_\mu)(p_\nu - P_\nu) + \sum_{\mu,\nu=n+1}^{n+K} \eta_{\mu\nu}[F^\mu - f^\mu(P, t_\mu)]\,[F^\nu - f^\nu(P, t_\nu)] \tag{16}$$

In this case,

$$
\begin{aligned}
f^\mu(x, t_\mu) &= x_\mu , & \mu &= 1, \cdots , n \\
a_i^\mu(x, t_\mu) &= \delta_{i\mu} & i &= 1, \cdots , n \\
& & \mu &= 1, \cdots , n \\
F^\mu &= p_\mu & \mu &= 1, \cdots , n
\end{aligned} \tag{17}
$$

Also

$$
\begin{aligned}
\rho_i(x) = &\sum_{j=1}^{n} \eta_{ij}(p_j - x_j) \\
&+ \sum_{\mu,\nu=n+1}^{n+K} \eta_{\mu\nu}[F^\mu - f^\mu(x, t_\mu]a_i^\nu(x, t_\nu) \\
&\qquad (i = 1, \cdots , n)
\end{aligned} \tag{18}
$$

and

$$
B_{ij}(x) = \eta_{ij} + \sum_{\mu,\nu=n+1}^{n+K} \eta_{\mu\nu}a_i^\nu(x, t_\nu)a_j^\mu(x, t_\mu) \qquad (i, j = 1, \cdots , n) \tag{19}
$$

If we also define $r^\mu(p, t_\mu)$ for $u = n + 1, \cdots , n + K$ by

$$
r^\mu(p, t_\mu) = F^\mu - f^\mu(p, t_\mu) \qquad \mu = n + 1, \cdots , n + K \tag{20}
$$

then $\rho_i(x)$ becomes, to first order,

$$
\begin{aligned}
\rho_i(x) = &\sum_{j=1}^{n} B_{ij}(x)(p_j - x_j) \\
&+ \sum_{\mu,\nu=n+1}^{n+K} \eta_{\mu\nu}a_i^\nu(x, t_\nu)r^\mu(p, t_\mu)
\end{aligned} \tag{21}
$$

Consequently, (13) reduces to

$$
P - x = p - x + [B(x)]^{-1}\rho^*(x) \tag{22}
$$

where

$$
\rho_i^*(x) = \sum_{\mu,\nu=n+1}^{n+K} \eta_{\mu\nu}a_i^\nu(x, t_\nu)r^\mu(p, t_\mu) \tag{23}
$$

Eq. (22) may be used as the basis for a first order differential correction to p, given the additional observations F^μ, $\mu = n + 1, \cdots , n + K$. First rewrite (22) as

$$
P - p = [B(x)]^{-1}\rho^*(x) \tag{24}
$$

Then, to first order, one can also write

$$
P - p = [B(p)]^{-1}\rho^*(p) \tag{25}
$$

Since all the quantities on the right of (25) are known, (25) gives the required first order correction to p.

A Stagewise Smoothing Procedure

Suppose the matrix $(\eta_{\mu\nu})$ can be written as a diagonal array of matrices

$$
\eta = \begin{pmatrix} \eta^{(1)} & & \\ & \ddots & \\ & & \eta^{(S)} \end{pmatrix} \tag{26}
$$

where $\eta^{(s)}$, $s = 1, \cdots , S$, are $N_s \times N_s$ matrices. For future notational convenience, we will regard the indices of the components $\eta_{\mu\nu}^{(s)}$ of $\eta^{(s)}$ as running over $\mu, \nu = M_{s-1} + 1, \cdots , M_s$ where

$$
\begin{aligned}
M_s &= \sum_{r=1}^{s} N_r \\
M_0 &= 0 \\
M_1 &= N_1 \geqq n \\
M_S &= N
\end{aligned} \tag{27}
$$

We define a stagewise smoothing procedure as follows: Suppose initial element estimates $\{P_i^{(1)}\}$ are obtained by minimization with respect to $P^{(1)}$ of the quadratic form

$$
\begin{aligned}
Q^{(1)} = \sum_{\mu,\nu=1}^{N_1} \eta_{\mu\nu}^{(1)}&[F^\mu - f^\mu(P^{(1)}, t_\mu)] \\
&\cdot [F^\nu - f^\nu(P^{(1)}, t_\nu)]
\end{aligned} \tag{28}
$$

Now define sequences of matrices $B^{(s)}$ and $B^{(s)}(x)$ as follows:*

$$
\begin{aligned}
B_{ij}^{(s)} &= B_{ij}^{(s-1)} \\
&\quad + \sum_{\mu,\nu=M_{s-1}+1}^{M_s} \eta_{\mu\nu}^{(s)} a_i^\nu(P^{(s)}, t_\nu)a_j^\mu(P^{(s)}, t_\mu) \\
B_{ij}^{(o)} &= 0 \\
B_{ij}^{(s)}(x) &= B_{ij}^{(s-1)}(x) \\
&\quad + \sum_{\mu,\nu=M_{s-1}+1}^{M_s} \eta_{\mu\nu}^{(s)} a_i^\nu(x, t_\nu)a_j^\mu(x, t_\mu) \\
B_{ij}^{(o)}(x) &= 0
\end{aligned} \tag{29}
$$

For $s > 1$, the s^{th} element estimates $\{P_i^{(s)}\}$ are obtained by minimizing, with respect to $P^{(s)}$, the quadratic form

$$
\begin{aligned}
Q^{(s)} = &\sum_{i,j=1}^{n} B_{ij}^{(s-1)}[P_i^{(s-1)} - P_i^{(s)}][P_j^{(s-1)} - P_j^{(s)}] \\
&+ \sum_{\mu,\nu=M_{s-1}+1}^{M_s} \eta_{\mu\nu}^{(s)}[F^\mu - f^\mu(P^{(s)}, t_\mu)] \\
&\qquad \cdot [F^\nu - f^\nu(P^{(s)}, t^\nu)]
\end{aligned} \tag{30}
$$

Let us also define $\rho_i^{(s)}(x)$, $s = 1, 2, \cdots , S$, by

$$
\begin{aligned}
\rho_i^{(s)}(x) = &\sum_{j=1}^{n} B_{ij}^{(s-1)}(x)[P_j^{(s-1)} - x_j] \\
&+ \sum_{\mu,\nu=M_{s-1}+1}^{M_s} \eta_{\mu\nu}^{(s)} a_i^\nu(x, t_\nu)[F^\mu - f^\mu(x, t_\mu)]
\end{aligned} \tag{31}
$$

Then, using (13), (18), and (19), it can be shown that to first order

$$
P^{(s)} - x = [B^{(s)}(x)]^{-1}\rho^{(s)}(x) \tag{32}
$$

* Alternative definitions for the matrices $B^{(s)}$ are possible see the remark at the end of this Section.

It is also not hard to show by an inductive argument that

$$P^{(S)} - x = [B^{(S)}(x)]^{-1}\rho^{(S)}(x) = [B(x)]^{-1}\rho(x) \quad (33)$$

where $B(x)$ and $\rho(x)$ are defined as in (9), (10), (11), (12). This is proved by showing that $B^{(S)}(x) = B(x)$, and $\rho^{(S)}(x) = \rho(x)$.

Thus, the first order dependence of the errors in the estimates $\{P_i^{(S)}\}$ is the same as that for the $\{P_i\}$ obtained by processing all N observations at once by minimizing Q as defined by (3). Another way of stating this is to say that, to first order, the estimate obtained by processing the N observations by the stagewise procedure just described is the same as that obtained by minimizing Q as defined by (3). (However, the range of magnitudes of $\mathcal{E}^\mu$ for which the first order expressions give good approximations to the estimation errors is not necessarily the same for the stagewise method as for the non-stagewise method.)

A stagewise smoothing procedure may be advantageous in certain situations. For example, suppose that observations are coming in at some average rate; in the stagewise procedure, it is not necessary to store all previous observations and $N \times N$ matrices $(\eta_{\mu\nu})$ with N increasing. It is at any time necessary to store only the 'current' element estimates $\{P_i^{(s)}\}$ and the matrix $B^{(s)}$, that is, $\frac{n}{2}(n + 3)$ quantities (taking into account the symmetry of $B^{(s)}$).

It can be seen that the matrices $B^{(s)}$ play the role of estimates of the matrices $B^{(s)}(x)$. Thus, the particular method of defining the sequence $\{B^{(s)}\}$ above is not the only one possible. For example, one could define

$$\begin{aligned}\tilde{B}_{ij}^{(s)} &= B_{ij}^{(s)}, \qquad s = 1\\ &= \tilde{B}_{ij}^{(s-1)} + \sum_{\mu,\nu=M_{s-1}+1}^{M_s} \eta_{\mu\nu}^{(s)} a_i^{\nu}(P^{(s-1)}, t_\nu) \qquad (29')\\ &\qquad \cdot a_j^{\mu}(P^{(s-1)}, t_\mu), \qquad s > 1\end{aligned}$$

and define the stagewise process using the matrices $\tilde{B}^{(s)}$ instead of $B^{(s)}$. Comparing with (29), we see that the main difference is that $\tilde{B}^{(s)}$ can be computed, for $s > 1$, before one computes $P^{(s)}$.

This lends itself conveniently to first order determination, for $s > 1$, of $P^{(s)} - P^{(s-1)}$ by means of Eq's. (20), (23), and (24). One would write $p = P^{(s-1)}$; $P = P^{(s)}$; and $P^{(s)} - P^{(s-1)} = [\tilde{B}^{(s)}]^{-1}\rho^{*(s)}(P^{(s-1)})$. (The vector $\rho^{*(s)}$ would be defined by obvious modifications of (20) and (23).)

Application when the Elements are Functions of Time

Suppose the elements $x_1, \cdots, x_n$ are functions of time: $x_i = x_i(t)$. Also suppose that error-free observations are given by

$$F^\mu = g^\mu[x(t_\mu), t_\mu] \quad (34)$$

Suppose the manner in which $x(t)$ depends on its values at some t_0 is known:

$$x_i(t) = \mathfrak{F}_i[x(t_0), t_0, t] \quad (35)$$

Then we may obtain the case considered in the previous sections by defining the elements x_i in the formulas of those sections to be $x_i = x_i(t_0)$ and by defining

$$f^\mu[x, t_\mu] = g^\mu\{\mathfrak{F}_1[x, t_0, t_\mu], \cdots, \mathfrak{F}_n[x, t_0, t_\mu], t_\mu\} \quad (36)$$

(where $x = x(t_0)$).

Modification when the Functions f^μ are Imperfectly Known

Suppose the observations are given by

$$F^\mu = h^\mu[x, t_\mu] + \mathcal{E}^\mu \quad (37)$$

(where $\mathcal{E}^\mu$ are observation errors),

but that the estimated elements $\{P_i\}$ are obtained by minimizing Q as defined by (3), with the functions f^μ (which differ from h^μ).

So long as $F^\mu - f^\mu$ are sufficiently small, equations (9)–(13) still describe the dependence of $P - x$ on $F^\mu - f^\mu$. This dependence was expressed

$$P_i - x_i = \sum_{\mu=1}^{N} \Gamma_i^\mu(x)[F^\mu - f^\mu(x, t_\mu)] \quad (14)$$

where:

$$\Gamma_i^\mu(x) = \sum_{j=1}^{n}\sum_{\nu=1}^{N} [B(x)]_{ij}^{-1}\eta_{\mu\nu}a_j^\nu(x, t_\nu) \quad (15)$$

The observation errors $\mathcal{E}^\mu$ are now given by

$$\mathcal{E}^\mu = F^\mu - h^\mu(x, t_\mu). \quad (38)$$

Therefore

$$F^\mu - f^\mu(x, t_\mu) = \mathcal{E}^\mu + h^\mu(x, t_\mu) - f^\mu(x, t_\mu) \quad (39)$$

Thus, (14) becomes

$$\begin{aligned}P_i - x_i = &\sum_{\mu=1}^{N} \Gamma_i^\mu(x)\mathcal{E}^\mu\\ &+ \sum_{\mu=1}^{N} \Gamma_i^\mu(x)[h^\mu(x, t_\mu) - f_\mu(x, t_\mu)]. \quad (40)\end{aligned}$$

Statistics of Propagated Errors

Eq. (13) may be used in an obvious manner to determine the means and covariance matrix of $\{P_i - x_i\}$ as functions of the means and covariance matrix of $\{\mathcal{E}^\mu\}$.

We shall deal here with a special case, namely, one in which the ensemble means and covariance matrix of $\{\mathcal{E}^\mu\}$ are known and in which the matrix $\{\eta_{\mu\nu}\}$ of the quadratic form Q has a special relation to the covariance matrix of $\{\mathcal{E}^\mu\}$.

Since the ensemble means of $\{\mathcal{E}^\mu\}$ are assumed known, it is no loss of generality to assume they are zero. In this case the covariance matrix is (denoting ensemble means by $E(\quad)$)

$$\phi_{\mu\nu} = E(\mathcal{E}^\mu\mathcal{E}^\nu) \quad (41)$$

It will now be assumed that the matrix $(\eta_{\mu\nu})$ in (3) is

$$(\eta_{\mu\nu}) = (\phi_{\mu\nu})^{-1} \quad \text{(matrix inverse)} \tag{42}$$

If $\{\mathcal{E}^\mu\}$ were to have a Gaussian probability distribution (and if $f^\mu \equiv h^\mu$, all μ) then the resulting method of obtaining P would be the maximum likelihood method.

The covariance matrix of $\{P_i - x_i\}$ assumes a particularly simple form in this case. For generality, we will deal with the case described in the last section, in which f^μ, the functions used in the quadratic form Q, may differ from h^μ.

The means of $\{P_i - x_i\}$ are

$$E(P_i - x_i) = \sum_{\mu=1}^{N} \Gamma_i^\mu(x)[h^\mu(x, t_\mu) - f^\mu(x, t_\mu)] \tag{43}$$

The covariance matrix of $\{P_i - x_i\}$ is readily established to be

$$\begin{aligned} E\{P_i - x_i - E(P_i - x_i)\} \\ \cdot \{P_j - x_j - E(P_j - x_j)\} = [B(x)]_{ij}^{-1} \end{aligned} \tag{44}$$

This holds whether the observations are processed all together or by a stagewise procedure as described. In the latter case, of course, it must be assumed that the covariance matrix of $\{\mathcal{E}^\mu\}$ can be broken up into a diagonal array of covariance matrices corresponding to the different stages. It is, in fact, quite easy to establish that $B^{(s)}(x)$ is the inverse covariance matrix of $\{P_i^{s)} - x_i\}$:

$$\begin{aligned} E\{P_i^{(s)} - x_i - E(P_i^{(s)} - x_i)\} \\ \cdot \{P_j^{s)} - x_j - E(P_j^{s)} - x_j)\} = [B^{(s)}(x)]_{ij}^{-1} \end{aligned} \tag{45}$$

This throws some further light on the stagewise method of Section II. The matrices $B^{(s-1)}$ occurring in the quadratic forms $Q^{(s)}$ are seen to be estimates of the inverse covariance matrices $B^{(s-1)}(x)$ of the element estimates $P^{(s-1)}$. Thus, if the error statistics were Gaussian, this procedure would consist at each stage of a maximum likelihood smoothing of the previous element estimates with the new observations.

The above formulas may be used to determine the rate at which the covariance matrix of the errors $P_i^{(s)} - x_i$ decreases as additional observations are processed. In fact, this information is contained in (44) and (45).

As a special case, consider the case where all observation errors are mutually uncorrelated. (If this is not true originally, it can be made true by means of linear transformations.) In this case, the matrices ϕ and η are diagonal.

If we now regard $P^{(s)}$ as the element estimate vector resulting from the processing of the first s observations, we have the inverse covariance matrix for $\{P_i^{(s)} - x_i\}$ given by

$$B_{ij}^{(s)}(x) = \sum_{\mu=1}^{s} \eta_{\mu\mu} a_i^\mu(x, t_\mu) a_j^\mu(x, t_\mu) \tag{46}$$

(This holds whether the s observations were processed all together or stagewise.)

It is also quite easy to verify that, for $s \geqq n$,

$$[B^{(s)}(x)]_{ij}^{-1} = [B^{(s-1)}(x)]_{ij}^{-1} - d_i^{(s)}(x)\, d_j^{(s)}(x) \tag{47}$$

where

$$d_i^{(s)}(x) = \frac{\sqrt{\eta_{ss}} \sum_{k=1}^{n} a_k^s(x, t_s)\, [B^{(s-1)}(x)]_{ik}^{-1}}{\sqrt{1 + \eta_{ss} \sum_{j,k=1}^{n} [B^{(s-1)}(x)]_{jk}^{-1} a_k^s(x, t_s)\, a_j^s(x, t_s)}} \tag{48}$$

We might close this section by briefly discussing the subject of systematic errors, which can be defined as errors which are highly correlated over a certain group of observations. There are several possible ways of handling systematic errors. They could be regarded as introducing off-diagonal elements into the correlation matrix $\phi_{\mu\nu}$ and the smoothing matrix $\eta_{\mu\nu}$, the magnitude of the off-diagonal elements being chosen according to the approximate magnitude of the systematic errors. Or, alternatively, a systematic error might in some cases profitably be regarded as an additional parameter to be estimated from the data—i.e., as an additional element. This is one of the choices that would have to be made in actual implementation of these results.

Modified Stagewise Procedure for Time-Varying Elements

In this section the elements will be considered functions of time, $x_i = x_i(t)$. It will be assumed that the elements vary much more slowly with respect to time than do the functions describing observations.

We suppose that the matrix $(\eta_{\mu\nu})$ can be written (as in Eq. (26)) as a diagonal array of matrices $\eta^{(s)}, s = 1, 2, \cdots$; the further assumption is made that the times t_μ, $\mu = M_{s-1} + 1, \cdots, M_s$, are sufficiently close together that they may be regarded as equal insofar as variation of the elements is concerned. This will be expressed

$$\begin{aligned} x_i(t_\mu) &= x_i(T_s) \\ \mu &= M_{s-1} + 1, \cdots, M_s \end{aligned} \tag{49}$$

Henceforth, the time parameter occurring in the argument of an element or element estimate will be written T.

It will be supposed that error-free observations are given by

$$\begin{aligned} F^\mu &= g^\mu[x(T_s), t_\mu] \\ \mu &= M_{s-1} + 1, \cdots, M_s \\ s &= 1, 2, \cdots \end{aligned} \tag{50}$$

and that the variation of the elements with respect to time is given by

$$x_i(T_s) = \mathfrak{F}_i[x(T_r), T_r, T_s]. \tag{51}$$

Now consider the following stagewise procedure for processing the observations (the main reason for which is its adaptability to cases in which the prediction functions f^μ or $\mathfrak{F}_i$ are imperfectly known):

The element estimate vector $P^+(T_1)$ is obtained by minimizing with respect to $P^+(T_1)$ the quadratic form

$$Q^{(1)} = \sum_{\mu,\nu=1}^{N_1} \eta_{\mu\nu}^{(1)}\{F^\mu - g^\mu[P^+(T_1), t_\mu]\} \cdot \{F^\nu - g^\nu[P^+(T_1), t_\nu]\}. \quad (52)$$

For $T_s \leqq T < T_{s+1}$, $s = 1, 2, \cdots$,

$$P_i(T) = \mathfrak{F}_i[P^+(T_s), T_s, T] \quad (53)$$

Also,

$$P_i^-(T_{s+1}) = \mathfrak{F}_i[P^+(T_s), T_s, T_{s+1}] \quad (54)$$

The quantities $P^+(T_s)$ will be defined for $s > 1$ below. To do this, we define sequences of matrices $B^{(s)}$, $B_+^{s)}$, $\psi^{(s)}$, $\psi_+^{s)}$ as follows:

$$B_{+ij}^{(1)} = \sum_{\mu,\nu=1}^{N_1} \eta_{\mu\nu}^{(1)} b_i^\nu[P^+(T_1), t_\nu] b_j^\mu[P^+(T_1), t_\mu] \quad (55)$$

$$b_i^\nu[x,t_\nu] = \frac{\partial g^\nu}{\partial x_i}[x,t_\nu] \quad (56)$$

$$\psi_+^{(s)} = [B_+^{(s)}]^{-1}, \qquad s = 1, 2, \cdots \quad (57)$$

$$\psi^{(s+1)} = \sum_{k,l=1}^{n} \mathfrak{A}_{ik}[P^+(T_s), T_s, T_{s+1}] \cdot \mathfrak{A}_{jl}[P^+(T_s), T_s, T_{s+1}]\psi_{+kl}^{(s)} \quad (58)$$

$$\mathfrak{A}_{ij}[x, T_s, T] = \frac{\partial \mathfrak{F}_i}{\partial x_j}[x, T_s, T] \quad (59)$$

$$\begin{aligned} B^{(s)} &= 0, \qquad s = 1 \\ &= [\psi^{(s)}]^{-1}, \qquad s > 1 \end{aligned} \quad (60)$$

$$B_{+ij}^{(s)} = B_{ij}^{(s)} + \sum_{\substack{\mu,\nu= \\ M_{s-1}+1}}^{M_s} \eta_{\mu\nu}^{(s)} b_i^\nu[P^+(T_s), t_\nu] b_j^\mu[P^+(T_s), t_\mu] \quad (61)$$

Then $P^+(T_s)$ is obtained for $s > 1$ by minimizing, with respect to $P^+(T_s)$, the quadratic form

$$\begin{aligned} Q^{(s)} = \sum_{i,j=1}^{n} & B_{ij}^{(s)}\{P_i^-(T_s) - P_i^+(T_s)\} \\ & \cdot\{P_j^-(T_s) - P_j^+(T_s)\} \\ & + \sum_{\substack{\mu,\nu= \\ M_{s-1}+1}}^{M_s} \eta_{\mu\nu}^{(s)}\{F^\mu - g^\mu[P^+(T_s), t_\mu]\} \\ & \cdot\{F^\nu - g^\nu[P^+(T_s), t_\nu]\} \end{aligned} \quad (62)$$

Scrutiny of Eqs. (52)–(62) reveals that a stagewise smoothing procedure has been completely defined. Now, in order to give the first order error equations for the resulting estimates, define the following matrices:

$$B_{+ij}[T_1] = \sum_{\mu,\nu=1}^{N_1} \eta_{\mu\nu}^{(1)} b_i^\nu[x(T_1), t_\nu] b_j^\mu[x(T_1), t_\mu] \quad (63)$$

(where $b_i^\nu(x, t_\nu)$ is defined as in (56));

$$\psi_+[T_s] = \{B_+[T_s]\}^{-1} \quad (64)$$

for $T_s < T \leqq T_{s+1}$,

$$\psi_{ij}[T] = \sum_{k,l=1}^{n} \mathfrak{A}_{ik}[x(T_s), T_s, T] \cdot \mathfrak{A}l_{jl}[x(T_s), T_s, T]\psi_{+kl}[T_s] \quad (65)$$

(where $\mathfrak{A}_{ij}(x, T_s, T)$ is defined as in (59));

$$B[T] = \{\psi[T]\}^{-1}, \; T_s < T \leqq T_{s+1} \quad (66)$$

and

$$B_{+ij}[T_s] = B_{ij}[T_s] + \sum_{\substack{\mu,\nu= \\ M_{s-1}+1}}^{M_s} \eta_{\mu\nu}^{(s)} b_i^\nu[x(T_s), t_\nu] b_j^\mu[x(T_s), t_\mu]. \quad (67)$$

Also define

$$\rho_i[T_s] = \sum_{j=1}^{n} B_{ij}[T_s][P_j^-(T_s) - x_j(T_s)] + \sum_{\substack{\mu,\nu= \\ M_{s-1}+1}}^{M_s} \eta_{\mu\nu}^{(s)} b_i^\nu[x(T_s), t_\nu]\{F^\mu - g^\mu[x(T_s), t_\mu]\}. \quad (68)$$

Then, the first order dependence of $P^+(T_s) - x(T_s)$ on the observation errors is:

$$P^+(T_s) - x(T_s) = \{B_+[T_s]\}^{-1}\rho[T_s] \quad (69)$$

Also, for $T_s < T < T_{s+1}$, the first order dependence of $P(T) - x(T)$ is

$$P_i(T) - x_i(T) = \sum_{j=1}^{n} \mathfrak{A}_{ij}[x(T_s), T_s, T][P_j^+(T_s) - x_j(T_s)] \quad (70)$$

Further, suppose that $(\eta_{\mu\nu})$ represents the inverse covariance matrix of $\{F^\mu - g^\mu[x(t_\mu), t_\mu]\}$. Then, $B_+[T_s]$ is the inverse covariance matrix of $\{P_i^+(T_s) - x_i(T_s)\}$, and $B[T]$ is the inverse covariance matrix of $\{P_i(T) - x_i(T)\}$, $T_s < T < T_{s+1}$.

In the third section an alternative method was described for dealing with time-varying elements. That method involved a reduction to the case of constant elements.

Choose any time T, $T_s \leqq T < T_{s+1}$, and regard T as now fixed. Define a constant element vector x by: $x = x(T)$. Let $P^*(T)$ represent the estimate of $x = x(T)$ obtained by the method described in the third section, based on the first M_s observations.

Then, it can be verified that, to first order, $P^*(T) = P(T)$, where $P(T)$ is defined by (53). That is, the stagewise procedure described in (52)–(62) yields an estimate $P(T)$ for $x(T)$ having the same first order dependence on $F^\mu - g^\mu[x(t_\mu), t_\mu]$, $\mu = 1, \cdots, M_s$, as the estimate derived by the method of Section III.

The stagewise procedure described above actually goes through the following steps: $P^+(T_s)$ represents an estimate of $x(T_s)$, based on the first M_s observations;

for $T_s < T < T_{s+1}$, $P(T)$ is obtained by (53), that is, simply by prediction from $P^+(T_s)$ according to the functional relation by which the elements are known to vary; $P^-(T_{s+1})$ represents the estimate of $x(T_{s+1})$ just before the observations F^μ, $\mu = M_s + 1, \cdots, M_{s+1}$ are processed.

The particular method chosen above to define the matrices $B_+^{(s)}$—i.e., by (61)—is not the only one possible. One could, for example, define matrices $\tilde{B}_+^{(s)}$ for $s > 1$ by using (61) with $P^-(T_s)$ instead of $P^+(T_s)$ in the arguments of b_i^ν and b_j^μ. Then one could compute $\tilde{B}_+^{(s)}$, for $s > 1$, before computing $P^+(T_s)$. This would be convenient for purposes of using (20), (23), and (24) for a first order determination of $P^+(T_s) - P^-(T_s)$: one would put $p = P^-(T_s)$; $P = P^+(T_s); P^+(T_s) - P^-(T_s) = [\tilde{B}_+^{(s)}]^{-1}\rho^{*(s)}[P^-(T)]$, with $\rho^{*(s)}$ defined by the appropriate modification of (20) and (23).

In practical applications, for example, in satellite observations where perturbation forces are imperfectly known, the functions $\mathfrak{F}_i$ may not be known exactly. Furthermore, the inaccuracy in knowledge of $\mathfrak{F}_i[x(T), T, T']$ will depend on the time difference $T' - T$—i.e., on how far ahead you are predicting the elements.

Suppose, for example, that the prediction functions used in the stagewise procedure are g^μ, $\mathfrak{F}_i$, but that the correct functions should be h^μ, $\mathfrak{F}_i^*$. Let $\mathcal{E}^\mu$ still represent the observation errors—i.e., $\mathcal{E}^\mu = F^\mu - h^\mu$—and let, for $T \geqq t_\mu$,

$$\begin{aligned}\delta^\mu(T) = h^\mu\{&\mathfrak{F}_1^*[x(T), T, t_\mu], \cdots, \\ &\mathfrak{F}_n^*[x(T), T, t_\mu], t_\mu\} \\ - g^\mu\{&\mathfrak{F}_1[x(T), T, t_\mu], \cdots, \\ &\mathfrak{F}_n[x(T), T, t_\mu], t_\mu\}\end{aligned} \tag{71}$$

Suppose that, in the case $g^\mu = h^\mu$, $\mathfrak{F}_i = \mathfrak{F}_i^*$, the first order error in $P(T)$, $T_s \leqq T < T_{s+1}$, could be expressed

$$P_i(T) - x_i(T) = \sum_{\mu=1}^{M_s} \Gamma_i^\mu(T)\mathcal{E}^\mu \tag{72}$$

Then, in the case $g^\mu \neq h^\mu$, $\mathfrak{F}_i \neq \mathfrak{F}_i^*$, one would have

$$P_i(T) - x_i(T) = \sum_{\mu=1}^{M_s} \Gamma_i^\mu(T)[\mathcal{E}^\mu + \delta^\mu(T)] \tag{73}$$

provided $\delta^\mu(T)$ and $\mathcal{E}^\mu$ are sufficiently small.

The errors $\delta^\mu(T)$ would, in general, grow as T increases. In such cases, it would be desirable to modify the smoothing procedure so as to give more recent observations greater weight, and continuously to diminish the weight given to past observations. There are a number of possibilities for accomplishing this one way, for instance, would be to define the matrices $B^{(s)}$ in (60)–(62) by

$$B_{ij}^{(s)} = [\psi^{(s)}]_{ij}^{-1}\lambda_i^{(s)}\lambda_j^{(s)} \tag{60*}$$

where $\lambda_i^{(s)}$ and $\lambda_j^{(s)}$ are $\leqq 1$.

This would mean that the effective smoothing matrix for the observations would not be the matrix (η) appearing in (52)–(62); in fact, there would really be no fixed smoothing matrix. Also, the first order error equations (63)–(70) would have to be modified appropriately.

There is an analogy between this class of smoothing procedures and the process of filtering of signals. One might regard the smoothing procedures as filters with input $x(T)$ and output $P(T)$. The case where the f^μ and $\mathfrak{F}_i$ are perfectly known is equivalent to a constant signal input to the filter, in which case the filter should have infinite memory to provide maximum smoothing of observation errors. If, say, $\mathfrak{F}_i$ are not exactly known, this corresponds to the existence of an unpredictable time varying component of the filter input; in this case the filter memory must be reduced (its 'bandwidth' increased); the best 'time constant' is related to the rate of variation of the unpredictable part of $x(T)$. One could also, in effect make the time constant different for the different elements.

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$$\frac{\cos(\pi x/2b)}{\cos(\pi\alpha/2b)}$$

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$(a + bx) \cos t$ is preferable to $\cos t\,(a + bx)$

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$$\{[a + (b + cx)^n] \cos ky\}^2$$

is required rather than $((a + (b + cx)^n) \cos ky)^2$.

Equations are numbered and referred to in text as (15).

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for periodicals—[2] Singer, S. F., "Artificial Modification of the Earth's Radiation Belt," *J. Astronaut. Sci.*, *6* (1959), 1–10.